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# FUCHSIAN AFFINE ACTIONS OF SURFACE GROUPS

FRANÇOIS LABOURIE


## Abstract

Let $\lambda_q$ be the irreducible representation of $SL(2,\mathbb{R})$ in $SL(q,\mathbb{R})$. Define a *Fuchsian subgroup* of $SL(q,\mathbb{R})$ to be a subgroup conjugate to a discrete subgroup of $\lambda_q(SL(2,\mathbb{R}))$. We prove in this paper that the fundamental group of a compact surface does not act properly on the affine space by affine tranformations if its linear part is Fuchsian.


## 1. Introduction

Let $\Gamma$ be the fundamental group of a compact surface. Let $\lambda_q$ be the irreducible $q$-dimensional representation of $SL(2,\mathbb{R})$ in $SL(q,\mathbb{R})$. We shall say a representation $\rho$ of $\Gamma$ in $SL(q,\mathbb{R})$ is *Fuchsian* (or *q-Fuchsian*) if $\rho = \lambda_q \circ \iota$, where $\iota$ is a discrete faithful representation of $\Gamma$ in $SL(2,\mathbb{R})$. We shall also say by extension the image of $\rho$ is *Fuchsian*, and that an affine action of a surface group is *Fuchsian*, if its linear part is Fuchsian.

Our main result is the following theorem:

**Theorem 1.1.** *A finite dimensional affine Fuchsian action of the fundamental group of a compact surface is not proper.*

In even dimensions, this is an easy remark, which has also been made in [2]. For dimension $4p + 1$, this theorem follows from the use of the *Margulis invariant* and Lemma 4.1, also due to Margulis (observation also made in [2]). This invariant and lemma were introduced in the work of Margulis [14] [15] in dimension 3, and later generalized in [16] [2] [9] [3] with his coauthors H. Abels and G. Soifer and also by T. Drumm in [9]. Therefore, our proof shall concentrate on dimensions $4p+3$ although we shall recall the proof in other dimensions in Section 6.

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This case bears special features: one should notice that G. Margulis has exhibited proper affine actions of a free group (with two generators) on $\mathbb{R}^3$ [14] [15], constructions later explained by T. Drumm in [5] [6] [8] and by V. Charette and W. Goldman in [4]. Therefore, surface groups behave differently than free groups in these dimensions.

When $\dim(E) = 3$, our result is a theorem of G. Mess [17], for which G. Margulis and W. Goldman [11] have obtained a different proof using Margulis invariant and Teichmüller theory. Our proof is based on similar ideas, but uses instead of Teichmüller theory a result on Anosov flows and a holomorphic interpretation of Margulis invariant, hence generalizing to higher dimensions.

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## 2. Representations of $SL(2,\mathbb{R})$, surfaces and connections

In this section, we describe the irreducible representation of $SL(2,\mathbb{R})$ of dimension $2n+1$ as the holonomy of a flat connection.

It is well known that in dimension 3, the irreducible representation of $SL(2,\mathbb{R})$ is associated with the Minkowski model of the hyperbolic plane $\mathbb{H}^2$. More precisely, there exists a flat connection on $E = \mathbb{R} \oplus T\mathbb{H}^2$, such that the action of $SL(2,\mathbb{R})$ lifts to a connection preserving action on this bundle. Hence, we obtain a 3-dimensional representation of $SL(2,\mathbb{R})$. Furthermore, the Minkowski model is obtained using the section $(1,0)$ of $E$.

We will now be more precise and explain this construction in more details in higher dimensions.

### 2.1 A flat connection

Let $\mathbb{H}^2$ be the oriented hyperbolic plane with its complex structure. Let $L_k$ be the complex line bundle over $\mathbb{H}^2$ defined by

$$L_k = (T\mathbb{H}^2)^{\otimes^k_{\mathbb{C}}}.$$

Let

$$E = \mathbb{R} \oplus L_1 \oplus \ldots \oplus L_n.$$

Notice now that $SL(2,\mathbb{R})$ acts on all $L_k$, hence on $E$, by bundle automorphisms.

If $Y$ is a section of $E$, $Y_0$ will denote its component on the factor $\mathbb{R}$, and $Y_k$ its component on $L_k$. The space of sections of the bundle $V$ will be denoted $\Gamma(V)$. The metric on $L_i$, induced from the Riemannian metric on $\mathbb{H}^2$ will be denoted $\langle\,,\rangle$. By definition, if $Y \in L_k$, $X \in L_1$, then $i_X Y$ is the element of $L_{k-1}$ such that

$$\forall Z \in L_k, \quad \langle i_X Y, Z\rangle = \langle Y, X \otimes Z\rangle.$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $L_1$, and, by extension, the induced connection on $L_k$. We introduce the following connection $\nabla$ on $E$, defined if $X \in T\mathbb{H}^2$, $Y \in \Gamma(E)$ by

$$\left\{\begin{array}{rcl} (\nabla_X Y)_0 & = & L_X Y_0 + \frac{1}{2}(n+1)\langle X, Y_1\rangle \\ \forall k>0,\ (\nabla_X Y)_k & = & (n-k+1)X \otimes Y_{k-1} + \overline{\nabla}_X Y_k \\ & & +\frac{1}{4}(n+k+1) i_X Y_{k+1}. \end{array}\right.$$

Consider the family or real numbers, defined for $k \in \{0, n-1\}$, by

$$a_0 = 1,\ a_{k+1} = \frac{1}{2^{2k+1}} \prod_{j=0}^{j=k} (\frac{n+j+1}{n-j}).$$

Define a metric of signature $(n, n+1)$ on $E$ by

$$\lfloor Y, Z \rfloor = \sum_{k=0}^{k=n} (-1)^{k+1} a_k \langle Y_k, Z_k\rangle.$$

The main result of this section is the following statement

**Proposition 2.1.** *The connection $\nabla$ is flat, and preserves the metric $\lfloor,\rfloor$. Furthermore, the $SL(2,\mathbb{R})$ action on $E$ preserves the metric $\lfloor,\rfloor$ and the connection $\nabla$. The resulting $(2n+1)$-representation of $SL(2,\mathbb{R})$ is irreducible.*

*Proof.* Long but straightforward computations (*cf.* Appendix A) show that $\nabla$ is flat, preserves $\lfloor,\rfloor$. Furthermore, the $SL(2,\mathbb{R})$ action on $E$ obviously preserves the metric $\lfloor,\rfloor$ and the connection $\nabla$.

We finally have to check that the corresponding representation of the group $SL(2,\mathbb{R})$ is irreducible. For that let $S^1 \subset SL(2,\mathbb{R})$, a subgroup isomorphic to the circle fixing a point $x_0$. The corresponding action on $L_k(x_0)$ is given by

$$e^{i\theta}(u) = e^{ki\theta} u.$$

This shows the representation is the irreducible $2n+1$ dimensional one. q.e.d.

## 3. Cohomology and holomorphic differentials

Let $S=\mathbb{H}^2/\Gamma$ be a compact surface. Let $\rho$ be a $(2n+1)$-Fuchsian representation of $\Gamma$. In this section, we shall describe the vector space $H^1_\rho(\Gamma,\mathbb{R}^{2n+1})$ in terms of holomorphic $(n+1)$-differentials on $S$.

We use the notations of the previous sections. Let $E_S=E/\Gamma$ be the vector bundle over $S=\mathbb{H}^2/\Gamma$ coming from $E$.

Let $\mathcal{H}^q$ the vector space of holomorphic $q$-differentials on $S$. Let $\Lambda^p(E_S)$ the vector space of $p$-forms on $S$ with value in $E_S$. The flat connection $\nabla$ gives rise to a complex

$$0\longrightarrow \Lambda^0(E)\xrightarrow{d^\nabla}\Lambda^1(E)\xrightarrow{d^\nabla}\Lambda^2(E)\longrightarrow 0.$$

The cohomology of this complex is $H^*_\rho(\Gamma,\mathbb{R}^{2n+1})$. From the metric on $\mathbb{H}^2$, we deduce an isomorphism $\omega\mapsto\breve{\omega}$ of $L_k^*$ with $L_k$. We define now a map $\Phi$ by

$$\Phi:\left\{\begin{array}{ccc}\mathcal{H}^{2n+1} & \to & \Lambda^1(E)\\ \omega & \mapsto & (X\mapsto i_X\breve{\omega}\in L_n\in E).\end{array}\right.$$

We first prove:

**Proposition 3.1.** *For every holomorphic $(n+1)$-differential $\omega$*

$$d^\nabla(\Phi(\omega))=0.$$

*Furthermore, if $\Phi(\omega)=d^\nabla u$, then $\omega=0$.*

*Proof.* By definition,

$$d^\nabla\Phi(\omega)(X,Y)=\nabla_X i_Y\breve{\omega}-\nabla_Y i_X\breve{\omega}-i_{[X,Y]}\breve{\omega}.$$

Hence, if $n>1$

$$d^\nabla\Phi(\omega)(X,Y)=\frac{2n}{4}(i_Xi_Y\breve{\omega}-i_Yi_X\breve{\omega})+(\overline{\nabla}_Xi_Y\breve{\omega}-\overline{\nabla}_Yi_X\breve{\omega}-i_{[X,Y]}\breve{\omega}).$$

Notice that $i_Xi_Y\breve{\omega}$ is symmetric in $X$ and $Y$. Finally, the holomorphicity condition on $\omega$ implies

$$\overline{\nabla}_Xi_Y\breve{\omega}-\overline{\nabla}_Yi_X\breve{\omega}-i_{[X,Y]}\breve{\omega}=0.$$

A similar proof (but with different constants) yields the result for $n=1$.

Next, assume $\Phi(\omega)=d^\nabla u$. The (non-Riemannian) metric on $E$ and the Riemannian metric on $\mathbb{H}^2$ induce a metric on $\Lambda^*(E)$, which we denote $\lfloor , \rfloor_\Lambda$. One should notice here that even though this metric is neither positive nor negative, since $\Phi(\omega)$ is a section of a bundle on which the metric is either positive or negative, we have

$$\lfloor \Phi(\omega), \Phi(\omega) \rfloor_\Lambda = 0 \Rightarrow \Phi(\omega) = 0 \Rightarrow \omega = 0.$$

Let $(d^\nabla)^*$ be the adjoint of $d^\nabla$. One has, if $(X_1, X_2)$ is a basis of $T\mathbb{H}^2$,

$$(d^\nabla)^*(\phi(\omega)) = -\sum_{k=1}^{k=2} \nabla_{X_k}(i_{X_k}\check{\omega}).$$

A short calculation shows

$$(d^\nabla)^*(\phi(\omega)) = -\sum_{k=1}^{k=2} \overline{\nabla}_{X_k}(i_{X_k}\check{\omega}),$$

and this last term is 0 by holomorphicity. We have just proved that

$$(d^\nabla)^*\Phi(\omega) = 0.$$

Hence, $\Phi(\omega) = d^\nabla u$ implies

$$\lfloor \Phi(\omega), \Phi(\omega) \rfloor_\Lambda = \lfloor (d^\nabla)^*\Phi(\omega), u \rfloor_\Lambda = 0.$$

This ends the proof. q.e.d.

It follows from the previous proposition that $\Phi$ gives rise to a map (also denoted $\Phi$) from $\mathcal{H}^{n+1}$ to the space $H^1_\rho(\Gamma, \mathbb{R}^{2n+1})$. We have:

**Corollary 3.2.** *The map $\Phi$ is an isomorphism from $\mathcal{H}^{n+1}$ to the space $H^1_\rho(\Gamma, \mathbb{R}^{2n+1})$.*

*Proof.* Indeed, we have just proved that $\Phi$ is injective. Furthermore, if $\chi(S)$ is the Euler characteristic of $S$, we have

$$\dim(H^1_\rho(\Gamma, \mathbb{R}^{2n+1})) \leq (2n+1)\chi(S).$$

But, by Riemann-Roch,

$$\dim(\mathcal{H}^{n+1}) = (2n+1)\chi(S).$$

Hence, the corollary follows. q.e.d.

### 3.1 Note added to the proof: Eichler-Shimura isomorphism

W. Goldman and the referee have both explained to me that the isomorphism between $\mathcal{H}^{n+1}$ and $H^1_\rho(\Gamma, \mathbb{R}^{2n+1})$ is a fairly well known instance of an *Eichler-Shimura isomorphism*. Indeed, let $V = \mathbb{R}^{2n+1}$, $\kappa$ be the canonical line bundle over $S$. Then $\mathcal{H}^{n+1}$ is the space $H^0(S; \mathfrak{o}(\kappa^{n+1}))$ of global holomorphic sections of $\kappa^{n+1}$. The global holomorphic sections of $\kappa^{-n}$ over $\mathbb{P}^1$ form a vector space isomorphic to $V \otimes \mathbb{C}$; this isomorphism is equivariant with respect to the natural actions of $SL(2, \mathbb{R})$. Thus, there is a sheaf homomorphism

$$V^* \longrightarrow \mathfrak{o}(\kappa^{-n}),$$

which defines a holomorphic section of

$$\kappa \otimes V \otimes \kappa^{-1-n},$$

and a cohomology class

$$Z_n \in H^1(S; V \otimes \kappa^{-1-n}).$$

Hence if $\omega \in H^0(S; \mathfrak{o}(\kappa^{n+1}))$, the product $\omega.Z_n$ belongs to $H^1(S, V)$. The *Eichler-Shimura isomorphism* is the map $\omega \mapsto \omega.Z_n$.

The original references to the Eichler-Shimura isomorphism (case n=1) are [10] [19] and a useful reference is [13].

This interpretation explains the isomorphism of 3.2, although the point of the construction made in this section is to have an explicit isomorphism at the level of forms in our setting.

## 4. A de Rham interpretation of Margulis invariant

The irreducible representation of $SL(2, \mathbb{R})$ of dimension $2n+1$ preserves a metric $\lfloor, \rfloor$ of signature $(n, n+1)$.

### 4.1 Loxodromic elements

We define a *loxodromic* element in $SO(n, n+1)$ to be $\mathbb{R}$-split and in the interior of a Weyl chamber. This just means all eigenvalues are real and have multiplicity 1. Recall that 1 always belong to the spectrum of a loxodromic element. Notice that all the elements, except the identity, of a $(2n+1)$-Fuchsian surface group are loxodromic.

## 4.2 The invariant vector of a loxodromic element

Chose now, once and for all, an orientation on $\mathbb{R}^{2n+1}$. The light cone - without the origin - has two components. Let's also choose one of these components.

Let $\gamma$ be a loxodromic element. It follows from the previous choices that we have a well defined eigenvector, the *invariant vector*, denoted $v_\gamma$, associated to the eigenvalue 1.

Indeed, all the other eigenvectors are lightlike. We order all the eigenvalues not equal to 1, in such a way that $\lambda_i < \lambda_{i+1}$. Thanks to our choices, we may pick one eigenvector $e_i$ in the preferred component of the light cone for all the eigenvalues $\lambda_i$ different than 1. We now choose $v_\gamma$ of norm 1, such that $(v_\gamma, e_1, \dots, e_{2n})$ is positively oriented.

## 4.3 Margulis invariant

Let $Iso(n, n+1) = \mathbb{R}^{2n+1} \rtimes SO(n, n+1)$ be the group of orientation preserving isometries of $\mathbb{R}^{2n+1}$ as an affine space. For $\gamma$ in $Iso(n, n+1)$, $\hat{\gamma}$ denotes its linear part. We shall say an element of $Iso(n, n+1)$ is *loxodromic* if its linear part is a loxodromic element of $SO(n, n+1)$.

The *Margulis invariant* ( [14] [15] ) of a loxodromic element $\gamma$ of $Iso(n, n+1)$ is

$$\mu(\gamma) = \lfloor \gamma(x) - x, v_{\hat{\gamma}} \rfloor,$$

where $x$ is an element of $\mathbb{R}^{2n+1}$. A quick check shows $\mu(\gamma)$ does not depend on $x$.

## 4.4 Margulis invariant and properness of an affine action

Let $\gamma_1$ and $\gamma_2$ be two loxodromic elements. Let $E_i^+$ (resp. $E_i^-$) be the space generated by the eigenvectors of $\hat{\gamma}_i$ corresponding to the eigenvalues of absolute value greater (resp. less) than 1.

We say $\gamma_1$ and $\gamma_2$ are in *general position* if the two decompositions

$$\mathbb{R}.v_{\hat{\gamma}_i} \oplus E_i^+ \oplus E_i^-,$$

are in general position.

Notice that for a $(2n+1)$-Fuchsian group, two (noncommensurable) elements are loxodromic and in general position.

In [14] [15] (see also [7]) G. Margulis has proved the following magic lemma:

**Lemma 4.1.** *If two loxodromic elements $\gamma_1$, $\gamma_2$, in general position, are such that $\mu(\gamma_1)\mu(\gamma_2) \leq 0$, then the group generated by $\gamma_1$ and $\gamma_2$ does not act properly on $\mathbb{R}^{2n+1}$.*

Drumm's articles [7], [9] as well as the survey by Abels [1] contain a more accessible and lucid proof of this lemma.

### 4.5 An interpretation of Margulis invariant

Let $\rho$ a representation of $\Gamma$ in $Iso(n, n+1)$, whose linear part, $\hat{\rho}$, is Fuchsian. Let $E_S = \mathbb{R} \oplus L_1 \oplus \ldots \oplus L_n$ the flat bundle over $S$ described in 2.1 whose holonomy is $\hat{\rho}$.

We describe now $\rho$ as an element of $H^1_{\hat{\rho}}(\Gamma, \mathbb{R}^{2n+1})$.

Let $\alpha \in H^1_{\hat{\rho}}(\Gamma, \mathbb{R}^{2n+1})$, interpreted as an element of $\Lambda^1(E_S)$. Let $\nabla^\alpha$ be the flat connection on $F = \mathbb{R} \oplus E_S$ defined by

$$\nabla^\alpha_X(\lambda, V) = (L_X\lambda, \lambda.\alpha(X) + \nabla_X V).$$

We claim there exists $\alpha \in H^1_{\hat{\rho}}(\Gamma, \mathbb{R}^{2n+1})$ such that the holonomy of $\nabla^\alpha$ is $\rho$. Of course, here, $\mathbb{R}^p \rtimes SL(p, \mathbb{R})$ is identified with a subgroup of $GL(p+1, \mathbb{R})$.

Let now $c$ be a closed curve on $S$, represented in homotopy by the conjugacy class of some element $\gamma$. Since $v_{\hat{\rho}(\gamma)}$ is invariant under $\hat{\rho}(\gamma)$, it gives rise to a parallel section $v_c$ of $E|_c$.

We first prove the following statement:

**Proposition 4.2.** *Let $c$, $\rho$, $\gamma$, $\alpha$ be as above. Then*

$$\mu(\rho(\gamma)) = \int_c \lfloor \alpha, v_c \rfloor.$$

*Proof.* We shall use the previous notations. We parametrise $c$ by the circle of length 1. Let $\pi$ be the covering $\mathbb{H}^2 \to S$. Consider a lift $\widetilde{c}$ of $c$ on the universal cover of $S$. The bundle $\pi^*F$ becomes trivial. The canonical section $\sigma$ corresponding to the $\mathbb{R}$ factor in $F$, gives rise to a map

$$i : \mathbb{H}^2 \to \mathbb{R}^{2n+1},$$

taking value in the affine hyperplane

$$P = \{(1, u) \in \mathbb{R}^{2n+1}\}.$$

Let $\bar{c} = i \circ \widetilde{c}$, and let's identify $\rho(\gamma)$ with $\gamma$. By definition now:

$$\begin{aligned}\mu(\gamma) &= \lfloor \rho(\gamma)(\bar{c}(0)) - \bar{c}(0), v_{\hat{\gamma}} \rfloor \\ &= \lfloor \bar{c}(1) - \bar{c}(0), v_{\hat{\gamma}} \rfloor \\ &= \int_0^1 \lfloor \dot{\bar{c}}(s), v_{\hat{\gamma}} \rfloor ds.\end{aligned}$$

Now, we interpret the last term on $F$ and we obtain

$$\begin{aligned}\mu(\gamma) &= \int_0^1 \lfloor \nabla^\alpha_{\dot{c}(s)} \sigma, v_c(s) \rfloor ds \\ &= \int_0^1 \lfloor \alpha(\dot{c}(s)), v_c(s) \rfloor ds \\ &= \int_c \lfloor \alpha, v_c \rfloor.\end{aligned}$$

This ends the proof. q.e.d.

## 4.6 The invariant vector as a section

In this subsection, we assume $n = 2p+1$, so that our representation is of dimension $4p+3$.

We use the notations of the previous subsections. In particular, let $\gamma \in \Gamma$. Let $v = v_{\hat{\rho}(\gamma)}$. Let $c$ be the closed geodesic (for the hyperbolic metric) corresponding to the element $\gamma$.

Recall that $v_\gamma$ gives rise to a section $v_c$ along the closed geodesic, which is parallel.

In this subsection, we wish to describe $v_c$ explicitly. Let $J$ the complex structure of $S$. Let's introduce the following section (along c) defined by

$$\begin{aligned}(w_c)_{2k} &= 0 \\ (w_c)_{2k+1} &= J(-4)^k \prod_{l=1}^{l=k} \left(\frac{p-l}{p+l+1}\right) \underbrace{\dot{c} \otimes \ldots \otimes \dot{c}}_{2k+1}.\end{aligned}$$

**Proposition 4.3.** *The section $w_c$ of $E_S$ is parallel along $c$. Furthermore, there exists $\varepsilon \in \{-1, 1\}$ independent of $c$ such that*

$$v_c = \varepsilon \frac{w_c}{\sqrt{\lfloor w_c, w_c \rfloor}}.$$

*Proof.* A straightforward computation shows that $w_c$ (hence $v_c$) is parallel. Furthermore $w_c$ is a spacelike vector, and by construction $v_c$ has norm 1.

It remains to prove that $v_c$ has the correct orientation. For that consider any geodesic arc $u$ in $\mathbb{H}^2$ parametrized by $[0, L]$. We have a basis of $E|_{u(t)}$ given by

$$B(t) = (1, \dot{u}, J\dot{u}, \dots, \underbrace{\dot{u} \otimes \dots \otimes \dot{u}}_{n}, J\underbrace{\dot{u} \otimes \dots \otimes \dot{u}}_{n}).$$

We may now consider the isometry $\gamma(u)$ sending $B(0)$ to $B(L)$. This is a loxodromic isometry. Next, consider the following section of $E$ along $u$ given by

$$\begin{aligned}(w_u)_{2k} &= 0\\ (w_u)_{2k+1} &= J(-4)^k \prod_{l=1}^{l=k} \left(\frac{p-l}{p+l+1}\right) \underbrace{\dot{u} \otimes \dots \otimes \dot{u}}_{2k+1}.\end{aligned}$$

This section is parallel along $u$ and therefore gives rise to a vector proportional to the invariant vector of $\gamma(u)$.

Next, by continuity, this proportion is constant. Applying this remark to a lift in the universal cover of our closed geodesic, this ends the proof. q.e.d.

## 5. Main theorem in dimension $4p + 3$

Again, let $\rho$ be a representation of a compact surface group $\Gamma$ in the group of affine transformations of an affine space of dimension $4p + 3$, whose linear part $\hat{\rho}$ is Fuchsian.

Notice first that for every nontrivial element $\gamma$ of $\Gamma$, $\hat{\rho}(\gamma)$ is loxodromic. Indeed, as an element of $SL(2, \mathbb{R})$, $\gamma$ is conjugate to a diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

where $\lambda < 1$. Now, the eigenvalues of $\hat{\rho}(\gamma)$, are those of its $2n$-th symmetric power:

$$\lambda^{2n} < \lambda^{2n-2} < \dots \lambda^2 < 1 < \lambda^{-2} < \dots < \lambda^{2-2n} < \lambda^{-2n}.$$

This proves $\hat{\rho}(\gamma)$ is loxodromic.

We assume now that $\rho(\Gamma)$ acts properly on $\mathbb{R}^{4p+3}$. The representation $\rho$ is described from $\hat{\rho}$ as an element $\alpha$ of $H^1_{\hat{\rho}}(\Gamma, \mathbb{R}^{4p+3})$.

According to Corollary 3.2, this element $\alpha$ is described by a holomorphic $(2p+2)$-differential $\omega$.

Let $\gamma \in \Gamma$, and $c$ the corresponding closed geodesic. From Proposition 4.2, we get

$$\mu(\rho(\gamma)) = \int_c \lfloor \alpha, v_c \rfloor.$$

From Proposition 4.3, we deduce there exists a constant $K_1$ just depending on $p$ such that

$$\mu(\rho(\gamma)) = K_1 \int_c \lfloor i_{\dot{c}}\omega, J\underbrace{\dot{c} \otimes \ldots \otimes \dot{c}}_{2p+1} \rfloor dt.$$

From the constructions explained in Section 2.1, we finally obtain there exists a constant $K_2$ just depending on $p$ such that

$$\begin{aligned} \mu(\rho(\gamma)) &= K_2 \int_c \langle i_{\dot{c}}\breve{\omega}, J\underbrace{\dot{c} \otimes \ldots \otimes \dot{c}}_{2p+1} \rangle dt \\ &= -K_2 \int_c \Im(\omega(\underbrace{\dot{c} \otimes \ldots \otimes \dot{c}}_{2p+2}))dt. \end{aligned}$$

Let $US$ be the unit tangent bundle of $S$. Let $f$ be the function defined on $US$ by

$$f(u) = \Im(\omega(\underbrace{u \otimes \ldots \otimes u}_{2p+2})).$$

From Lemma 4.1 and the previous computation, we obtain that the integral of $f$ along closed orbits of the geodesic flow has a constant sign. On the other hand, let $\lambda$ be the Lebesgue measure, we have

$$\int_{US} f d\lambda = 0.$$

Indeed, let $\beta$ be a complex number such that $\beta^{2p+2} = -1$. Scalar multiplication by $\beta$ defines a diffeomorphism of $US$, which preserves both the orientation and the Lebesgue measure. Lastly $f \circ \beta = -f$ and this proves the last formula.

The conclusion of the proof follows at once from the following lemma, since the Lebesgue measure for the geodesic flow of a constant curvature surface is the Bowen-Margulis measure.

**Lemma 5.1.** *Let $M$ be a compact manifold equipped with a topologically transitive Anosov flow. Let $\nu$ be the Bowen-Margulis measure. Let $f$ be a Hölder function defined on $M$ such that its integral on every closed orbit is positive, then the integral of $f$ with respect to $\nu$ is positive.*

I could not find a proper reference in the literature of this specific lemma, for which I claim no originality. G. Margulis has suggested to use a central limit theorem of M. Ratner [18]. We shall rather explain a proof using the notions of pressure and equilibrium states.

*Proof.* Let's denote by $\mathcal{M}$ the space of invariant probability measures. For any $\mu$ in $\mathcal{M}$, $h(\mu)$ will be its entropy. Recall that the Bowen-Margulis measure maximizes the entropy. Next define the *pressure* of a Hölder function $f$ by

$$P(f) = \sup_{\mu\in\mathcal{M}} \big(h(\mu) + \int_M f d\mu\big).$$

By definition, an invariant measure $\mu$ is called an *equilibrium state* for $f$, if $P(f) = h(\mu) + \int_M f d\mu$. Hence, the Bowen-Margulis measure is an equilibrium state for the zero function.

Thanks to results of Bowen, every Hölder function admits a unique equilibrium state. This is stated as Theorem 20.3.7 in [12]. Furthermore, according to Proposition 20.3.10 of [12], two Hölder continuous functions with the same equilibrium state are equal up to the addition of a constant and a coboundary. More precisely, this result is stated in the case of diffeomorphisms but the proof generalizes for flows.

Now, let $f$ be as in the lemma. Assume that $\int_M f d\nu = 0$ and let's look for a contradiction. Recall that for a topologically transitive Anosov flow, as a consequence of the shadowing lemma, every invariant measure is a weak limit of barycenters of measures supported on closed orbits [20]. Hence

$$\forall \mu \in \mathcal{M}, \int_M f d\mu \geq 0.$$

It follows that

$$P(-f) = P(0) = h(\nu) = h(\nu) + \int_M (-f d\nu).$$

Hence, the zero function and $-f$ have the same equilibrium state $\nu$. It follows from the above discussion that $f$ is a cohomologous to a constant. On one hand, this constant is zero, since $\int_M f d\nu = 0$. On the other hand, this constant is nonzero because integrals of $f$ over closed orbits are positive. Here is our contradiction. Actually, the proof would work for any measure in the measure class of a Gibbs measure, although we shall not need it. q.e.d.

## 6. Other dimensions

The other dimensions are either easy (even case) or follows from the immediate use of Margulis invariant ($4p+1$ case) as we shall explain now. Similar arguments can be found in [2].

Let $\lambda$ be the representation of $SL(2,\mathbb{R})$ of even dimension. Let $h$ be a loxodromic element of $SL(2,\mathbb{R})$. Then 1 will not belong to the spectrum of $\lambda(h)$. It follows, that if $\rho$ is a representation of $\Gamma$ in even dimension whose linear part is Fuchsian then for all $\gamma$ in $\Gamma$ not equal to the identity then $\rho(\gamma)$ does not act properly.

Last, in dimensions $4p+1$, the Margulis invariant is such that $\mu(\gamma^{-1}) = -\mu(\gamma)$. It follows at once from Lemma 4.1, that if $\rho$ is a representation of $\Gamma$ in dimension $4p+1$ whose linear part is Fuchsian, if $\gamma_1$ and $\gamma_2$ are noncommensurable elements of $\Gamma$ then $\rho(\gamma_1)$ and $\rho(\gamma_2)$ generate a group that does not act properly on the affine space. Of course, the point in our previous discussion is that in dimension $4p+3$ then $\mu(\gamma^{-1}) = \mu(\gamma)$, hence such an argument do not work and actually, free groups (even Fuchsian ones) can act properly, see [14], [15] and [7].

## Appendix A: some computations

We explain here how to make the computations delayed from the proof of Proposition 2.1. Let $X$, $Z$ two commuting vector fields on $\mathbb{H}^2$. Let $\omega$ the Kähler form of $\mathbb{H}^2$ defined by $\omega(Z,X) = \langle JZ, X\rangle$. Let's first introduce the following notation. If $f$ is a function of $Z$ and $X$ then

$$\underline{f(Z,X)} = f(Z,X) - f(X,Z).$$

With these notations at hands, we have

$$\underline{Z \otimes \langle X, Y\rangle} = \frac{1}{2}\big(\underline{Z\otimes i_X Y}\big) = -\frac{1}{2}\big(\underline{i_Z(X\otimes Y)}\big) = -\omega(Z,X)JY.$$

Let $\overline{R}$ be the curvature tensor of $\overline{\nabla}$ and recall that

$$\overline{R}(Z,X)Y_k = k\omega(Z,X)JY.$$

Let $R$ be the curvature tensor of $\nabla$. We first have

$$\begin{aligned}(R(Z,X)Y)_0 =& +\frac{1}{2}(n+1)(n)\underline{\langle Z,X\rangle\otimes Y_0}\\ &+\frac{1}{8}(n+1)(n+2)\underline{\langle Z,i_XY_2\rangle}\\ =&0.\end{aligned}$$

Next

$$\begin{aligned}(R(Z,X)Y)_1 =& +\frac{1}{2}n(n+1)\underline{Z\otimes\langle X,Y_1\rangle}+\overline{R}(Z,X)Y_1\\ &+\frac{1}{4}(n+2)(n-1)\underline{i_Z(X\otimes Y_1)}\\ &+\frac{1}{16}(n+2)(n+3)\underline{i_Zi_XY_3}\\ =&\omega(Z,X)JY\big(-\frac{1}{2}n(n+1)+1+\frac{2}{4}(n+2)(n-1)\big)\\ =&0.\end{aligned}$$

It remains to consider the case $k>1$. We get

$$\begin{aligned}(R(Z,X)Y)_k =&\overline{R}(Z,X)Y_k+(n-k+1)(n-k+2)\underline{(Z\otimes X\otimes Y_{k-2})}\\ &+\frac{1}{4}\big((n-k+1)(n+k)\underline{Z\otimes i_XY_k}\\ &+(n+k+1)(n-k)\underline{i_Z(X\otimes Y_k)}\big)\\ &+\frac{1}{16}(n+k+1)(n+k+2)\underline{i_Zi_XY_{k+2}}.\end{aligned}$$

Hence

$$\begin{aligned}&(R(Z,X)Y)_k\\ &\quad=\omega(Z,X)JY\big(-\frac{2}{4}(n-k+1)(n+k)+k+\frac{2}{4}(n+k+1)(n-k)\big)\\ &\quad=0.\end{aligned}$$

We have just proved the connection $\nabla$ is flat. Now, we show $\nabla$ preserves

$\lfloor , \rfloor$. Let $Y$ a section of $E$. Then

$$\begin{aligned}
\lfloor \nabla_X Y, Y \rfloor =& \sum_{k=0}^{k=n} (-1)^{k+1} a_k \langle (\nabla_X) Y_k, Y_k \rangle \\
=& - \langle L_X Y_0, Y_0 \rangle + \sum_{k=1}^{k=n} (-1)^{k+1} a_k \langle \overline{\nabla}_X Y_k, Y_k \rangle \\
& - \frac{1}{2}(n+1) \langle \langle X, Y_1 \rangle, Y_0 \rangle \\
& + (-1)^{k+1} \sum_{k=1}^{k=n} a_k \frac{(n+k+1)}{4} \langle i_X Y_{k+1}, Y_k \rangle \\
& + (-1)^{k+1} \sum_{k=1}^{k=n} a_k (n-k+1) \langle X \otimes Y_{k-1}, Y_k \rangle .
\end{aligned}$$

We make a change of variables in the last term, and get

$$\begin{aligned}
\lfloor \nabla_X Y, Y \rfloor =& L_X \lfloor Y, Y \rfloor - \frac{1}{2}(n+1) \langle \langle X, Y_1 \rangle, Y_0 \rangle \\
& + n a_1 \langle X \otimes Y_0, Y_1 \rangle + \sum_{k=1}^{k=n} (-1)^k \big( a_{k+1}(n-k) \\
& - a_k \frac{(n+k+1)}{4} \big) \langle X \otimes Y_k, Y_{k+1} \rangle .
\end{aligned}$$

To conclude, we just have to remark that

$$a_1 = \frac{n+1}{2n}, \ \frac{a_{k+1}}{a_k} = \frac{n+k+1}{4(n-k)}.$$

## References


[1] H. Abels, *Properly discontinuous groups of affine transformations, A survey,* Preprint.

[2] H. Abels, G. Margulis & G. Soifer, *Properly discontinuous groups of affine transformations with orthogonal linear part,* C. R. Acad. Sci. Paris Sr. I Math. **324** (1997) 253–258.

[3] ———, *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations,* SFB Bielefeld Preprint 97–083.

[4] V. Charette & W. Goldman, *Affine Schottky groups and crooked tilings,* to appear in Proc. workshop Crystallographic Groups and their Generalizations II, in Kortrijk, Belgium, Contemp. Math., Amer. Math. Soc.

[5] T. Drumm & W. Goldman, *Complete flat Lorentz 3-manifolds with free fundamental group,* Int. J. Math. **1** (1990) 149–161.

[6] T. Drumm, *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992) 677–691.

[7] ———, *Examples of non proper affine actions,* Mich. Math. J. **30** (1992), 435-442.

[8] ———, *Linear holonomy of Margulis space-times,* J. Differential Geom. **38** (1993) 679–691.

[9] ———, *Translations and the holonomy of complete affine flat manifolds,* Math. Res. Lett. **1** (1994) 757–764.

[10] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale,* Math. Z. **67** (1957) 267–298.

[11] W. Goldman & G. Margulis, *Flat Lorentz 3-manifolds and cocompact Fuchsian groups,* to appear in Proc. workshop Crystallographic Groups and their Generalizations II, in Kortrijk, Belgium, Contemp. Math., Amer. Math. Soc.

[12] B. Hasselblatt & A. Katok, *Introduction to the modern theory of dynamical systems,* Encyclopedia of Mathematics and Its Applications **54**, Cambridge Univ. Press, Cambridge, 1997.

[13] S. Katok & J. Millson, *Eichler-Shimura homology, intersection numbers and rational structures on spaces of modular forms,* Trans. Amer. Math. Soc. **300** (1987) 737–757.

[14] G. Margulis, *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk SSSR **272** (1985) 181–193.

[15] ———, *Complete affine locally flat manifolds with a free fundamental group,* J. Soviet. Math. **134** (1987) 129–131.

[16] ———, *On the Zariski closure of the linear Part of a properly discontinuous group of affine transformations,* Preprint, 1987.

[17] G. Mess, *Lorentz spacetimes of constant curvature,* Inst. Hautes Études Sci, Preprint, 1990.

[18] M. Ratner, *The central limit theorem for geodesic flows on n-dimensional manifolds of negative curvature,* Israel J. Math. **16** (1973) 181–197.

[19] G. Shimura, *Sur les intégrales attachées aux formes automorphes,* J. Math. Soc. Japan **11** (1959) 291–311.

[20] K. Sigmund, *On the space of invariant measures for hyperbolic flows,* Amer. J. Math. **94** (1972) 31–37.

Université Paris-Sud